Chapter 6 Logarithmic and Exponential Functions

1. Find the exact solution of $3^{2x} - 3^{x+1} - 4 = 0$.

[4]

Let $y = 3^x$

$y^2 - 3y - 4 = 0$

$(y-4)(y+1) = 0$

$y = 4$ or $y = -1$

$3^x = 4$ $\quad$ $3^x = -1$ (reject)

$x \lg 3 = \lg 4$

$x = \log_3 4$

2. $h(x) = 2ln(3x - 1)$ for $x \geq \frac{2}{3}$.

The graph of $y = h(x)$ intersects the line *y* = *x* at two distinct points. On the axes below, sketch the graph of $y = h(x)$ and hence sketch the graph of $y = h^{-1}(x)$

[4]

$y=0$, $0 = 2\ln(3x-1)$

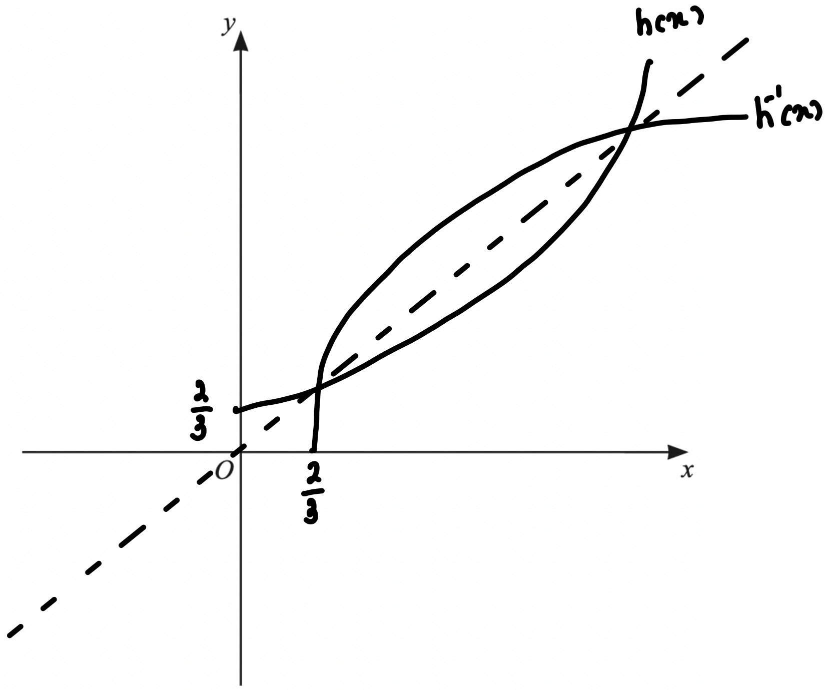
$0 = \ln(3x-1)$

$e^0 = 3x-1$

$1 = 3x-1$

$2 = 3x$

$x = \frac{2}{3}$



3. (a) Given that $log_2 x + 2log_4 y = 8$, find the value of *xy*.

[3]

$\log_2 x + 2\log_{2^2} y = 8$

$\log_2 x + \log_2 y = 8$

$\log_2 xy = 8$

$xy = 2^8 = 256$

(b) Using the substitution $y = 2^x$, or otherwise, solve $2^{2x+1} - 2^{x+1} - 2^x + 1 = 0$.

[4]

$2 \quad -1 \quad 1$

$1 \quad -1 \quad 2$

$2y^2 - 2y - y + 1 = 0$

$2y^2 - 3y + 1 = 0$

$(2y - 1)(y - 1) = 0$

$y = \frac{1}{2}$ or $y = 1$

$2^x = \frac{1}{2}$ $\qquad$ $2^x = 1$

$2^x = 2^{-1}$ $\qquad$ $2^x = 2^0$

$x = -1$ $\qquad$ $x = 0$

4. (a) Solve the simultaneous equations

$$10^{x+2y} = 5,$$

$$10^{3x+4y} = 50,$$

giving *x* and *y* in exact simplified form.

[4]

$x + 2y = \lg 5 \quad \times 2$

$3x + 4y = \lg 50$

$- \; 2x + 4y = 2\lg 5$

$x = \lg 50 - \lg 25$

$x = \lg 2$

$2y = \lg 5 - \lg 2$

$2y = \lg \frac{5}{2}$

$y = \frac{1}{2} \lg \frac{5}{2}$

(b) Solve $2x^{\frac{2}{3}} - x^{\frac{1}{3}} - 10 = 0.$

[3]

$2y^2 - y - 10 = 0$

2 - 5 5

1 + 2 4

$(2y-5)(y+2) = 0$

$y = \frac{5}{2}$ or $y = -2$

$x^{1/3} = \frac{5}{2}$ $\quad x^{1/3} = -2$

$x = \frac{125}{8}$ $\quad x = -8$

5. $log_a\sqrt{b} - \frac{1}{2} = log_b a$, where $a > 0$ and $b > 0$.

Solve this equation for b, giving your answers in terms of a. [5]

$\frac{1}{2}\log_a b - \lg_b a = \frac{1}{2}$

$\frac{1}{2}\lg_a b - \frac{1}{\lg_a b} = \frac{1}{2}$

let $y = \lg_a b$

$\frac{1}{2}y - \frac{1}{y} = \frac{1}{2} \quad \times 2$

$y - \frac{2}{y} = 1$

-2, $+1$

$y^2 - 2 - y = 0$

$(y-2)(y+1) = 0$

$y = 2$ or $y = -1$

$\log_a b = 2$ | $\log_a b = -1$

$b = a^2$ | $b = a^{-1}$

$b = \frac{1}{a}$

6. Solve the simultaneous equations.

$$log_3(x + y) = 2$$

$$2log_3(x + 1) = log_3(y + 2)$$

[6]

$x + y = 3^2$

$x + y = 9$ — ① $\leftarrow y = 9 - x$

$(x+1)^2 = (y+2)$

$x^2 + 2x + 1 = y + 2$

$x^2 + 2x + 1 = 9 - x + 2$

$x^2 + 3x + 1 - 11 = 0$

$x^2 + 3x - 10 = 0$

$+5$, -2

$(x+5)(x-2) = 0$

$x = -5$ or $x = 2$

$y = 14$ | $y = 7$

7. **DO NOT USE A CALCULATOR IN THIS QUESTION.**

$$log_2(y + 1) = 3 - 2log_2x$$

$$log_2(x + 2) = 2 + log_2y$$

a. Show that $x^3 + 6x^2 - 32 = 0$. [4]

$\log_2(y+1) + \log_2 x^2 = 3$

$x^2(y+1) = 8$ — ①

$\log_2(x+2) - \log_2 y = 2$

$\frac{x+2}{y} = 4$

$y = \frac{x+2}{4}$ — ②

$x^2\left(\frac{x+2}{4} + 1\right) = 8$

$x^2\left(\frac{x+2+4}{4}\right) = 8$

$x^2\left(\frac{x+6}{4}\right) = 8$

$x^3 + 6x^2 = 32$

$x^3 + 6x^2 - 32 = 0$

(shown)

b. Find the roots of $x^3 + 6x^2 - 32 = 0$. [4]

Let $f(x) = x^3 + 6x^2 - 32$

$f(2) = 8 + 24 - 32$

$= 0$

$(x-2)$ is a factor of $f(x)$.

$$
\begin{array}{r|l}
 & x^2 + 8x + 16 \\
\hline
x-2 & x^3 + 6x^2 + 0x - 32 \\
 & x^3 - 2x^2 \\
\hline
 & 8x^2 + 0x \\
 & 8x^2 - 16x \\
\hline
 & 16x - 32 \\
 & 16x - 32 \\
\hline
 & 0
\end{array}
$$

$f(x) = (x-2)(x^2+8x+16)$

$= (x-2)(x+4)(x+4)$

$x = 2$ or $x = -4$

c. Give a reason why only one root is a valid solution of the logarithmic equations. Find the value of *y* corresponding to this root. [2]

$x = 2$ is the only valid solution since lg cannot take negative value.

$y = \frac{x+2}{4} = 1$

8. It is given that $f(x) = 5ln(2x + 3)$ for $x > -\frac{3}{2}$.

a. Write down the range of f. [1]

$y \in \mathbb{R}$

b. Find f^{-1} and state its domain. [3]

$f(x) = 5\ln(2x+3)$

$y = 5\ln(2x+3)$

$\frac{y}{5} = \ln(2x+3)$

$e^{y/5} = 2x+3$

$\frac{e^{y/5} - 3}{2} = x$

$f^{-1}(x) = \frac{e^{x/5} - 3}{2}, \ x \in \mathbb{R}$

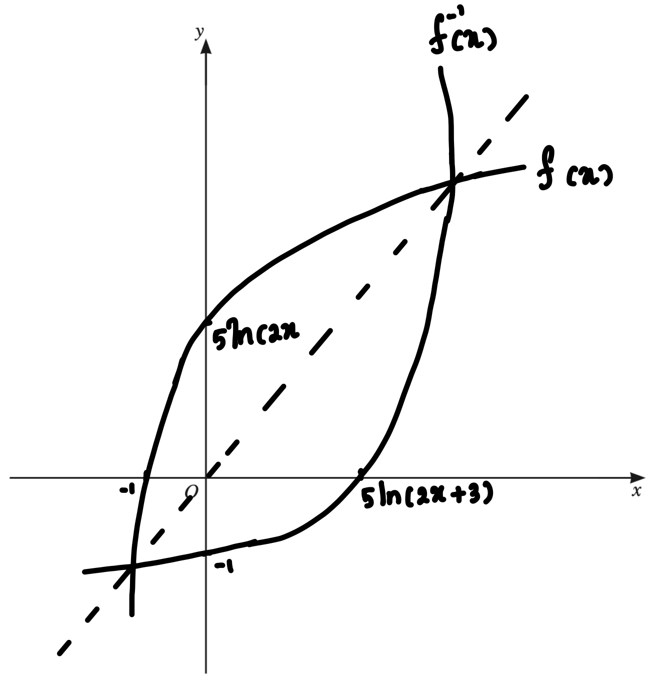
c. On the axes below, sketch the graph of $y = f(x)$ and the graph of $y = f^{-1}(x)$. Label each curve and state the intercepts on the coordinate axes. [5]

$y = 5\ln(2x+3)$

$x = 0, \ y = 5\ln 3$

$y = 0, \ x = \frac{1-3}{2}$

$= -1$



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9. $f(x) = 4ln\,(2x - 1)$

a. Write down the largest possible domain for the function f.

[1]

$2x - 1 > 0$

$2x > 1$

$x > \frac{1}{2}$

b. Find $f^{-1}(x)$ and its domain.

[3]

$y = 4\ln(2x-1)$

$\frac{y}{4} = \ln(2x-1)$

$e^{y/4} = 2x - 1$

$\frac{e^{y/4}+1}{2} = x$

$f^{-1}(x) = \frac{e^{\frac{x}{4}}+1}{2}, x \in \mathbb{R}$

10. Write $3lg\,x + 2 - lg\,y$ as a single logarithm.

[3]

$\lg\frac{x^3}{y} + 2\lg 10$

$\lg\frac{x^3}{y} + \lg 100$

$\lg\frac{100x^3}{y}$

11. The population *P*, in millions, of a country is given by $P = A \times b^{t}$, where *t* is the number of years after January 2000 and *A* and *b* are constants. In January 2010 the population was 40 million and had increased to 45 million by January 2013.

a. Show that *b* = 1.04 to 2 decimal places and find *A* to the nearest integer. [4]

$P = A \times b^{t}$

$40 = A \times b^{10}$

$A = \frac{40}{b^{10}}$

$45 = A \times b^{13}$

$A = \frac{45}{b^{13}}$

$\frac{40}{b^{10}} = \frac{45}{b^{13}}$

$b^{3} = \frac{45}{40}$

$b^{3} = \frac{9}{8}$

$\therefore b = 1.040$

$b \approx 1.04$

$A = \frac{45}{(1.04)^{13}}$

$= 27.02$

≈ 27

b. Find the population in January 2020, giving your answer to the nearest million. [1]

$P = 27 \times 1.04^{t}$

$= 27 \times 1.04^{20}$

$= 59.16$

≈ 59 million

c. In January of which year will the population be over 100 million for the first time? [3]

$100 = 27 \times 1.04^{t}$

$\frac{100}{27} = 1.04^{t}$

$\lg \frac{100}{27} = t \lg 1.04$

$t = 33.38$

≈ 34

$\therefore 2034$

12. The number, b, of bacteria in a sample is given by $b = P + Qe^{2t}$, where P and Q are constants and t is time in weeks. Initially there are 500 bacteria which increase to 600 after 1 week.

a. Find the value of P and of Q.

[4]

$$b = P + Qe^{2t}$$

$$500 = P + Qe^{0}$$

$$P + Q = 500 \quad -①$$

$$600 = P + Qe^{2}$$

$$-\ 500 = -P + -Q$$

$$100 = Qe^{2} - Q$$

$$100 = Q(e^{2} - 1)$$

$$Q = 15.7$$

$$P = 500 - 15.7$$

$$= 484.3$$

b. Find the number of bacteria present after 2 weeks.

[1]

$$b = P + Qe^{2t}$$

$$= 484.3 + 15.7e^{4}$$

$$= 1341.49$$

c. Find the first week in which the number of bacteria is greater than 1000000.

[3]

$$1000000 = 484.3 + 15.7e^{2t}$$

$$63663.42038 = e^{2t}$$

$$11.061 = 2t$$

$$t = 5.53$$

$$t = 6 \text{ week}$$